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OBSERVATION ON THE TERNARY QUADRATIC DIOPHANTINE EQUATION WITH THREE UNKNOWNS $13x^2+3y^2=640z^2$

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Abstract:

The ternary quadratic equation given by $13x^2+3y^2=640z^2$ is considered and searched for its many different integer solution. Seven different choices of integer solution of the above equations are presented. A few interesting relations between the solutions and special polynomial numbers are presented.

Keywords: Ternary quadratic, integer solutions

Notation:

$$t_{m,n}=n\left[1+\frac{(n-1)(m-2)}{2}\right]$$

$$PR_n=n(n+1)$$

$$G_n=2n-1$$

INTRODUCTION :

The Diophantine equations offer an unlimited field for research due to their variety [1-3]. In particular , one may refer [4-12] for quadratic equations with three unknowns. This communication concerns with yet another interesting equation $13x^2+3y^2=640z^2$ representing homogeneous equation with three unknowns for determining its infinitely many non-zero integral points. Also,few interesting relations among the solutions are presented.

METHOD OF ANALYSIS

The Quadratic Diophantine equation with three unknowns to be solved is given by,

$$13x^2 + 3y^2 = 640z^2 \tag{1}$$

Consider the linear transformation

$$\left.\begin{aligned} x &= X - 3T \\ y &= X + 13T \end{aligned}\right\} \tag{2}$$

Substituting (2) in (1) we get,

$$X^2 + 39T^2 = 40Z^2 \tag{3}$$

PATTERN: 1

Assume $z = a^2 + 39b^2$ (4)

Write 40 as,

$$40 = \left(1 + i\sqrt{39}\right)\left(1 - i\sqrt{39}\right) \tag{5}$$

Substituting (4) ,(5) in (3) and employing the method of factorization, we get

$$\left(X + i\sqrt{39}T\right)\left(X - i\sqrt{39}T\right) = \left(1 + i\sqrt{39}\right)\left(1 - i\sqrt{39}\right)\left(a + i\sqrt{39}b\right)^2\left(a - i\sqrt{39}b\right)^2$$

Equating the positive factor,

$$\left(X + i\sqrt{39}T\right) = \left(1 + i\sqrt{39}\right)\left(a + ib\sqrt{39}b\right)^2$$

$$\Rightarrow \left(X + i\sqrt{39}T\right) = \left(a^2 - 78ab - 39b^2\right) + i\sqrt{39}\left(a^2 + 2ab - 39b^2\right)$$

Equating real and imaginary parts,

$$X = a^2 - 78ab - 39b^2$$

$$T = a^2 + 2ab - 39b^2$$

Substituting in (2)

$$x(a,b) = -2a^2 + 78b^2 - 84ab$$

$$y(a,b) = 14a^2 - 546b^2 - 52ab$$

The non-zero distinct integer solution of (1) is obtained as

$$x(a,b) = -2a^2 + 78b^2 - 84ab$$

$$y(a,b) = 14a^2 - 546b^2 - 52ab$$

$$z(a,b) = a^2 + 39b^2$$

PROPERTIES:

- $.x(a,1) + y(a,1) - 148t_{4,a} + 136P_{ra} \equiv 0 \pmod{2}$
- $.x(a,1) + z(a,1) - 83t_{4,a} - 84P_{ra} \equiv 0 \pmod{3}$
- $z(a,1) + y(a,1) - 67t_{4,a} - 52P_{ra} - 50 = 0$
- $y(a,1) - 14z(a,1) \equiv 0 \pmod{52}$
- $7x(a,1) + y(a,1) + G_{320a} + 1 = 0$

PATTERN:2

'40' can also be written as

$$40 = \frac{(11 + i\sqrt{39})(11 - i\sqrt{39})}{2^2} \tag{6}$$

Substituting (4) ,(5) &(6) in (3) and employing the method of factorization, we get

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$$\left(X+i\sqrt{39}T\right)\left(X-i\sqrt{39}T\right)=\frac{\left(11+i\sqrt{39}\right)\left(11-i\sqrt{39}\right)\left(a+i\sqrt{39}b\right)^{2}\left(a-i\sqrt{39}b\right)^{2}}{2^{2}}$$

Equating the positive factor

$$\left(X+i\sqrt{39}T\right)=\frac{\left(11+i\sqrt{39}\right)\left(a+i\sqrt{39}b\right)^{2}}{2}$$

$$\Rightarrow \quad \left(X+i\sqrt{39}T\right)=\left(11a^{2}-78ab-429b^{2}\right)+i\sqrt{39}\left(a^{2}+22ab-39b^{2}\right)$$

Equating real and imaginary parts of the above equation, we get

$$X=\frac{1}{2}\left(11a^{2}-78ab-429b^{2}\right)$$

$$T=\frac{1}{2}\left(a^{2}+22ab-39b^{2}\right)$$

Substituting in (2)

$$x=4a^{2}-72ab-156b^{2}$$

$$y=12a^{2}+104ab-468b^{2}$$

The non- zero distinct integral solution of (1) is obtained as

$$x(a,b)=4a^{2}-72ab-156b^{2}$$

$$y(a,b)=12a^{2}+104ab-468b^{2}$$

$$z(a,b)=a^{2}+39b^{2}$$

PROPERTIES:

- $x(a,1)+y(a,1)+16t_{4,a}-32P_{ra}\equiv 0 \pmod{2}$
- $x(a,1)+z(a,1)-85t_{4,a}+72P_{ra}\equiv 0 \pmod{3}$
- $y(a,1)+z(a,1)+91t_{4,a}-104P_{ra}+429=0$

- $x(a,1) - 4z(a,1) \equiv 48 (\bmod 72)$

- $3x(a,1) - y(a,1) + G_{160a} + 1 = 0$

PATTERN: 3

Write the equation (3) as

$$X^2 + 39T^2 = 40Z^2 \times 1 \tag{7}$$

'1' can be written as

$$1 = \frac{(7 + 3i\sqrt{39})(7 - 3i\sqrt{39})}{20^2} \tag{8}$$

Substituting (6), (7) and (8) in (3) and employing the method of factorization, we get

$$(X + i\sqrt{39}T)(X - i\sqrt{39}T) = \frac{(7 + 3i\sqrt{39})(7 - 3i\sqrt{39})}{20^2} \frac{(11 + i\sqrt{39})(11 - i\sqrt{39})}{2^2} (a + i\sqrt{39}b)^2 (a - i\sqrt{39}b)^2$$

Equating the positive factor

$$(X + i\sqrt{39}T) = \frac{(7 + 3i\sqrt{39})}{20} \frac{(11 + i\sqrt{39})}{2} (a + i\sqrt{39}b)^2$$

$$\Rightarrow (X + i\sqrt{39}T) = \frac{(-40a^2 + 1560b^2 + 3120ab) + i\sqrt{39}(40a^2 - 1560b^2 - 80ab)}{40}$$

Equating real and imaginary part of the above equation, we get

$$X = -a^2 + 39b^2 + 78ab$$

$$T = -a^2 - 39b^2 - 2ab$$

Substituting in (2)

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$$x=-4a^2+156b^2-72ab$$

$$y=12a^2-468b^2-104ab$$

The non-zero distinct integral solution of (1) is obtained as

$$x(a,b)=-4a^2+156b^2-72ab$$

$$y(a,b)=12a^2-468b^2-104ab$$

$$z(a,b)=a^2+39b^2$$

PROPERTIES:

- $x(a,1)+y(a,1)-184t_{4,a}+352P_{ra}+312=0$
- $.x(a,1)+z(a,1)-69t_{4,a}+72P_{ra}\equiv 0(\bmod 5)$
- $.y(a,1)+z(a,1)-117t_{4,a}+104P_{ra}\equiv 0(\bmod 3)$
- $3x(a,1)+y(a,1)+G_{160a}+1=0$
- $y(a,1)-12z(a,1)\equiv 0(\bmod 104)$

PATTERN:4

(3) can also be written as

$$(X+Z)(X-Z)=39(Z+T)(Z-T) \tag{9}$$

Case: 1

(9) can be written in the form of ratio as

$$\frac{(X+Z)}{13(Z+T)}=\frac{3(Z-T)}{(X-Z)}=\frac{\alpha}{\beta},\beta\neq 0 \tag{10}$$

which is equivalent to the system of double equation as

$$\left.\begin{aligned}X\beta-13\alpha T+Z(\beta-13\alpha)=0\\ -X\alpha-3\beta T+Z(\alpha+3\beta)=0\end{aligned}\right\} \tag{11}$$

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Solving (11) by the method of cross multiplication, we get

$$\left.\begin{aligned} X &= -13\alpha^2 - 78\alpha\beta + 3\beta^2 \\ T &= -13\alpha^2 + 2\alpha\beta + 3\beta^2 \\ z &= -13\alpha^2 - 3\beta^2 \end{aligned}\right\} \qquad (12)$$

Substituting (12) in (2) the non-zero distinct integer solution of (1) are given by

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$$x(\alpha,\beta) = 26\alpha^2 - 84\alpha\beta - 6\beta^2$$

$$y(\alpha,\beta) = -182\alpha^2 - 52\alpha\beta + 42\beta^2$$

$$z(\alpha,\beta) = -13\alpha^2 - 3\beta^2$$

PROPERTIES:

- $x(\alpha,1) + y(\alpha,1) + 20t_{4,a} + 136P_{ra} \equiv 0 \pmod{2}$
- $x(\alpha,1) + y(\alpha,1) - 97_{4,a} + 84P_{ra} \equiv 0 \pmod{3}$
- $y(\alpha,1) + z(\alpha,1) + 143_{4,a} + 52P_{ra} - 1 = 0$
- $x(\alpha,1) + 2z(\alpha,1) \equiv 7 \pmod{84}$
- $7x(\alpha,1) + y(\alpha,1) + G_{320\alpha} + 1 = 0$

Case: 2

(9) can also be written in the form of the ratio as

$$\frac{X+Z}{3(Z+T)} = \frac{13(Z-T)}{(X-Z)} = \frac{\alpha}{\beta}, \beta \neq 0$$

which is equivalent to the system of double equation as

$$\left.\begin{aligned} X\beta - 3\alpha T + Z(-3\alpha + \beta) &= 0 \\ -X\alpha - 13\beta T + Z(13\beta + \alpha) &= 0 \end{aligned}\right\} \tag{13}$$

Solving (13) by the method of cross multiplication, we get

$$\left.\begin{aligned} X &= -3\alpha^2 - 78\alpha\beta + 13\beta^2 \\ T &= -3\alpha^2 + 2\alpha\beta + 13\beta^2 \\ z &= -3\alpha^2 - 13\beta^2 \end{aligned}\right\} \tag{14}$$

substituting (14) in (2), the non-zero distinct integral solution of (1) are given by

$$x(\alpha,\beta) = 6\alpha^2 - 84\alpha\beta - 26\beta^2$$

$$y(\alpha,\beta) = -42\alpha^2 - 52\alpha\beta + 182\beta^2$$

$$z(\alpha,\beta) = -3\alpha^2 - 13\beta^2$$

PROPERTIES:

- $x(\alpha,1) + y(\alpha,1) - 100t_{4,a} + 136P_{r,a} \equiv 0(\mathrm{mod}\,2)$
- $.x(\alpha,1) + z(\alpha,1) - 87t_{4,a} - 84P_{r,a} + 23 = 0$
- $.y(\alpha,1) + z(\alpha,1) - 7t_{4,a} + 52_{r,a} \equiv 0(\mathrm{mod}\,5)$
- $x(\alpha,1) + 2z(\alpha,1) \equiv 52(\mathrm{mod}\,84)$
- $7x(\alpha,1) + y(\alpha,1) + G_{320\alpha} + 1 = 0$

Case: 3

(9) can be written in the form of the ratio as

$$\frac{X+Z}{39(Z+T)}=\frac{(Z-T)}{X-Z}=\frac{\alpha}{\beta},\beta\neq 0$$

which is equivalent to the system of double equation as

$$\left.\begin{aligned} &X\beta-39\alpha T+Z(-39\alpha+\beta)=0 \\ &-\alpha X-\beta T+Z(\alpha+\beta)=0 \end{aligned}\right\} \tag{15}$$

solving (15) by the method of cross multiplication, we get

$$\left.\begin{aligned} &X=-39\alpha^2-78\alpha\beta+\beta^2 \\ &T=-39\alpha^2+2\alpha\beta+\beta^2 \\ &z=-39\alpha^2-\beta^2 \end{aligned}\right\} \tag{16}$$

substituting (16) in (2), we obtained the non-zero distinct integral solution of (1) are given by

$$x(\alpha,\beta)=78\alpha^2-84\alpha\beta-2\beta^2$$

$$y(\alpha,\beta)=-546\alpha^2-52\alpha\beta+14\beta^2$$

$$z(\alpha,\beta)=-39\alpha^2-\beta^2$$

PROPERTIES:

- $x(\alpha,1)+y(\alpha,1)+332t_{4,a}+136P_{r,a}\equiv 0(\mathrm{mod}\,3)$
- $x(\alpha,1)+z(\alpha,1)-123t_{4,a}+84P_{r,a}\equiv 0(\mathrm{mod}\,3)$
- $.y(\alpha,1)+z(\alpha,1)+533t_{4,a}+52P_{r,a}-13=0$
- $y(\alpha,1)-14z(\alpha,1)\equiv 24(\mathrm{mod}\,52)$
- $7x(\alpha,1)+y(\alpha,1)+G_{320\alpha}+1=0$

Case:4

(9) can be written in the form of the ratio as

$$\frac{X+Z}{Z+T}=\frac{39(Z-T)}{X-Z}=\frac{\alpha}{\beta},\beta\neq 0 \qquad (17)$$

which is equivalent to the system of double equation as

$$\left.\begin{aligned} X\beta-\alpha T+Z(-\alpha+\beta)=0 \\ -\alpha X-39\beta T+Z(\alpha+39\beta)=0 \end{aligned}\right\} \qquad (18)$$

Solving (18) by the method of cross multiplication, we get

$$\left.\begin{aligned} X &= -\alpha^2-78\alpha\beta+39\beta^2 \\ T &= -\alpha^2+2\alpha\beta+39\beta^2 \\ z &= -\alpha^2-39\beta^2 \end{aligned}\right\} \qquad (19)$$

Substituting (19) in (1), the non-zero distinct integral solution of (1) are given by

$$x(\alpha,\beta)=2\alpha^2-84\alpha\beta-78\beta^2$$

$$y(\alpha,\beta)=-14\alpha^2-52\alpha\beta+546\beta^2$$

$$z(\alpha,\beta)=-\alpha^2-39\beta^2$$

PROPERTIES:

- $.x(\alpha,1)+y(\alpha,1)-124t_{4,a}-136P_{ra}\equiv 0(\bmod 2)$
- $x(\alpha,1)+z(\alpha,1)-85t_{4,a}+84P_{ra}+117=0$
- $y(\alpha,1)+z(\alpha,1)-37t_{4,a}+52P_{ra}\equiv 0(\bmod 3)$
- $y(\alpha,1)-14z(\alpha,1)\equiv 0(\bmod 52)$

- $7x(\alpha,1)-14z(\alpha,1)+G_{320\alpha}+1=0$

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Conclusion:

In this paper, an attempt has been made to obtain non-zero distinct integer solutions to the ternary quadratic diophantine equation representing $13x^2+3y^2=640z^2$ homogenous cone. As the Diophantine equations are rich in variety, one may search for integer solutions to higher degree Diophantine equations with multiple variables along with suitable properties.

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